# Estimating Distances from Quadruples Satisfying Stability Properties to Quadruples not Satisfying Them

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**Abstract.-** We consider quadruples of matrices $(E, A, B, C)$ defining generalized linear multivariable time-invariant dynamical systems $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with $A, E$ square matrices and $B, C$ rectangular matrices. Using geometrical techniques we present upper bounds and lower bounds for the distances between a quadruple and the nearest structurally unstable, uncontrollable and/or unobservable one, in terms of the singular values of matrices associated to the quadruple.

## 1. Introduction

We consider generalized linear finite-dimensional time-invariant dynamical systems given by differential-algebraic equations (DAE's)

$$\left.\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\}$$

where $E, A \in M_{r\times n}(\mathbb{F}), B \in M_{r\times m}(\mathbb{F}), C \in M_{p\times n}(\mathbb{F})$ and $\mathbb{F}$ is the field of real or complex numbers.

These equations arise in theoretical areas as Differential Equations on manifolds, as well as in applied areas as in Control Theory. They are obtained when modelling different set-ups,

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for instance, when modelling mechanical multibody systems and electrical circuits (see [GP83], [GR93], [Ho95], [Ra96], [Si94],...). Several authors, like Mehrmann, Kunkel, etc. (see, for example, [KM94]) have widely studied these equations.

We will assume in all the paper that $r = n$. This assumption does not suppose a restriction to our problem, since in the case $r < n$ it suffices to add $n - r$ rows to matrices $E, A, B$ with zero entries and, in the case $r > n$, it suffices to add $r - n$ columns to matrices $E, A, C$ with zero entries, thus obtaining in both cases a system with the same set of solutions (in the case $r > n$, $r - n$ state variables have been added).

We will consider in the set of quadruples of matrices $(E, A, B, C)$ the equivalence relations corresponding to one, or more, of the following standard transformations in the set of dynamical systems defined by the quadruples: basis changes in the state, control and output spaces, state feedback, derivative feedback and output injection.

We are interested in obtaining upper and lower bounds for the distances between a quadruple of matrices satisfying a property and the nearest quadruple not satisfying it. The properties we will deal with are: structural stability, controllability and/or observability. They all have a deep interest in Control Theory.

Several authors, as Boley, Lu, Eising , Kägstrom, Demmel, Edelman, etc. (see [Bo90], [BL86], [Ei84], [DE85], [EEK97], for example) analyze bounds for the distance between pairs of matrices or matrix pencils to the nearest pair or matrix pencil with qualitative different properties.

The structure of the paper is as follows.

In Section §2 we introduce the equivalence relations in the space of quadruples of matrices which are suitable for our goals and view them as those induced by the actions of Lie groups.

In Section §3, a geometrical study of orbits and tangent spaces to the orbits is made.

Sections §4, §5 and §6 are devoted to recall the usual matrix norms and define the distance between two quadruples of matrices, to recall the concepts of controllable and/or observable systems and the matritial characterizations in terms of the controllability and observability matrices associated to a set of matrices defining the system, and to recall the concept of structural stability, as appears in [Wi80], respectively.

In Section §7, we obtain a lower bound for the distance between a structurally stable quadruple and the nearest non-structurally stable one with respect to different equivalence relations.

In Section §8 we measure the distance between a controllable and observable quadruple of matrices and the nearest uncontrollable or/and unobservable one. An upper bound is obtained in terms of the singular values of the controllability, observability and controllability-observability matrices associated to the quadruple, realizing a similar study to that in [BL86].

Finally, in Section §9, some examples are presented, and the the bounds obtained in the preceding Sections are discussed.

## 2. Equivalence relations and Lie group actions

We will denote by $\mathbb{F}$ a commutative field. Let us consider the set

$$\mathcal{Q}(\mathbb{F}) = \{(E, A, B, C) \mid E, A \in M_n(\mathbb{F}), B \in M_{n\times m}(\mathbb{F}), C \in M_{p\times n}(\mathbb{F})\}$$

of quadruples of matrices defining a DAE. We consider the following standard transformations in $\mathcal{Q}(\mathbb{F})$:

(1) basis similarity for the state space: $(E, A, B, C) \longrightarrow (P^{-1}EP, P^{-1}AP, P^{-1}B, CP)$;

(2) basis changes for the control space: $(E, A, B, C) \longrightarrow (E, A, BR, C)$;

(3) basis changes for the output space: $(E, A, B, C) \longrightarrow (E, A, B, SC)$;

(4) output injection: $(E, A, B, C) \longrightarrow (E, A + TC, B, C)$;

(5) state feedback: $(E, A, B, C) \longrightarrow (E, A + BU, B, C)$;

(6) derivative feedback: $(E, A, B, C) \longrightarrow (E + BV, A, B, C)$;

for some matrices $P \in Gl_n(\mathbb{F})$, $R \in Gl_m(\mathbb{F})$, $S \in Gl_p(\mathbb{F})$, $T \in M_{n\times p}(\mathbb{F})$ and $U, V \in M_{m\times n}(\mathbb{F})$.

This leads to the definition of the following equivalence relation in the space $\mathcal{Q}(\mathbb{F})$.

**Definition 1.** Two quadruples $(E_1, A_1, B_1, C_1)$, $(E_2, A_2, B_2, C_2)$ are feedback-equivalent if, and only if, there exist matrices $P \in Gl_n(\mathbb{F})$, $R \in Gl_m(\mathbb{F})$, $S \in Gl_p(\mathbb{F})$, $T \in M_{n\times p}(\mathbb{F})$ and $U, V \in M_{m\times n}(\mathbb{F})$ such that

$$E_2 = P^{-1}E_1P + P^{-1}B_1V, \; A_2 = P^{-1}A_1P + TC_1P + P^{-1}B_1U, \; B_2 = P^{-1}B_1R, \; C_2 = SC_1P$$

We will make use of the following notation: $(E_1, A_1, B_1, C_1) \sim_f (E_2, A_2, B_2, C_2)$.

Let us consider now the linear varieties of $\mathcal{Q}(\mathbb{F})$:

$$\mathcal{V}_1(\mathbb{F}) = (I_n, 0, 0, 0) + \mathcal{Q}_1(\mathbb{F}), \mathcal{Q}_1(\mathbb{F}) = \{(0, A, B, C) \mid A \in M_n(\mathbb{F}), B \in M_{n\times m}(\mathbb{F}), C \in M_{p\times n}(\mathbb{F})\}$$

$$\mathcal{V}_2(\mathbb{F}) = (I_n, 0, 0, 0) + \mathcal{Q}_2(\mathbb{F}), \mathcal{Q}_2(\mathbb{F}) = \{(0, A, B, 0) \mid A \in M_n(\mathbb{F}), B \in M_{n\times m}(\mathbb{F})\}$$

$$\mathcal{V}_3(\mathbb{F}) = (I_n, 0, 0, 0) + \mathcal{Q}_3(\mathbb{F}), \mathcal{Q}_3(\mathbb{F}) = \{(0, A, 0, C) \mid A \in M_n(\mathbb{F}), C \in M_{p\times n}(\mathbb{F})\}$$

We will consider the equivalence classes in these linear varieties with respect to the following equivalence relations.

**Definition 2.** Two quadruples $(I_n, A_1, B_1, C_1)$, $(I_n, A_2, B_2, C_2)$ in $\mathcal{V}_1(\mathbb{F})$ are called similar if, and only if, there exists $P \in Gl_n(\mathbb{F})$ such that

$$A_2 = P^{-1}A_1P, \quad B_2 = P^{-1}B_1, \quad C_2 = C_1P$$

That is to say, when the triple $(A_2, B_2, C_2)$ may be obtained from $(A_1, B_1, C_1)$ by means of the elementary transformation (1).

**Definition 3.** Two quadruples $(I_n, A_1, B_1, 0)$, $(I_n, A_2, B_2, 0)$ in $\mathcal{V}_2(\mathbb{F})$ are called block-similar if, and only if, there exist matrices $P \in Gl_n(\mathbb{F})$, $R \in Gl_m(\mathbb{F})$ and $U \in M_{m\times n}(\mathbb{F})$ such that

$$A_2 \;= P^{-1}A_1P + P^{-1}B_1U, \quad B_2 = P^{-1}B_1R$$

That is to say, when the pair $(A_2, B_2)$ may be obtained from $(A_1, B_1)$ by means of one, or more, of the following elementary transformations: (1), (2) and (5).

**Definition 4.** Two quadruples $(I_n, A_1, 0, C_1)$, $(I_n, A_2, 0, C_2)$ in $\mathcal{V}_3(\mathbb{F})$ are called left block-similar if, and only if, there exist matrices $P \in Gl_n(\mathbb{F})$, $S \in Gl_p(\mathbb{F})$ and $T \in M_{n\times p}(\mathbb{F})$ such that

$$A_2 \;= P^{-1}A_1P + TC_1P, \quad C_2 = SC_1P$$

That is to say, when the pair $(A_2, C_2)$ may be obtained from $(A_1, C_1)$ by means of one, or more, of the following elementary transformations: (1), (3) and (4).

We will make use of the following notation: $(I_n, A_1, B_1, C_1)\sim_s(I_n, A_2, B_2, C_2)$, $(I_n, A_1, B_1, 0)\sim_b(I_n, A_2, B_2, 0)$, $(I_n, A_1, 0, C_1)\sim_l(I_n, A_2, 0, C_2)$, respectively.

**ASSUMPTION:** From now on, $\mathbb{F}$ will denote the field of real or complex numbers.

The equivalence relations defined in $\mathcal{Q}(\mathbb{F})$, $\mathcal{V}_1(\mathbb{F})$, $\mathcal{V}_2(\mathbb{F})$ and $\mathcal{V}_3(\mathbb{F})$ can be viewed as those induced by Lie group actions on the respective subjacent vector spaces.

Concretely, we can consider the action $\alpha$ on $\mathcal{Q}(\mathbb{F})$ of the Lie group

$$\mathcal{G}(\mathbb{F}) = Gl_n(\mathbb{F}) \times Gl_m(\mathbb{F}) \times Gl_p(\mathbb{F}) \times M_{n\times p}(\mathbb{F}) \times M_{m\times n}(\mathbb{F}) \times M_{m\times n}(\mathbb{F})$$

where the product is defined by

$$(P_1, R_1, S_1, T_1, U_1, V_1)\circ(P_2, R_2, S_2, T_2, U_2, V_2) = (P_2P_1, R_2R_1, S_1S_2, P_1^{-1}T_2{+}T_1S_2, U_2P_1{+}R_2U_1, V_2P_1{+}R_2V_1)$$

with identity element $I = (I_n, I_m, I_p, 0, 0, 0)$ and the inverse element of $(P, R, S, T, U, V)$ being $(P^{-1}, R^{-1}, S^{-1}, -PTS^{-1}, -R^{-1}UP^{-1}, -R^{-1}VP^{-1})$. This Lie group acts on $\mathcal{Q}(\mathbb{F})$ as follows,

$$\begin{aligned} \alpha : \mathcal{G}(\mathbb{F}) \times \mathcal{Q}(\mathbb{F}) \;&\longrightarrow \mathcal{Q}(\mathbb{F}) \\ ((P, R, S, T, U, V), (E, A, B, C)) \;&\longrightarrow (P^{-1}EP + P^{-1}BV, P^{-1}AP + TCP + P^{-1}BU, P^{-1}BR, SCP) \end{aligned}$$

Any equivalence class coincides with the orbit of any quadruple in it under this action. For any quadruple $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$, we will denote by $\mathcal{O}(E, A, B, C)$ the orbit of this quadruple under the action $\alpha$.

Note that $\mathcal{V}_1(\mathbb{F})$, $\mathcal{V}_2(\mathbb{F})$, $\mathcal{V}_3(\mathbb{F})$, $\mathcal{Q}_1(\mathbb{F})$ and $\mathcal{Q}_2(\mathbb{F})$ are not invariant under the action $\alpha$; that is to say, $\alpha(\mathcal{G}(\mathbb{F}), \mathcal{V}_i(\mathbb{F}))$ is not included in $\mathcal{V}_i(\mathbb{F})$, $i = 1, 2, 3$, and $\alpha(\mathcal{G}(\mathbb{F}), \mathcal{Q}_j(\mathbb{F}))$ is not included in $\mathcal{Q}_j(\mathbb{F})$, $j = 1, 2$.

We will view now the equivalence relations in $\mathcal{Q}_1(\mathbb{F})$, $\mathcal{Q}_2(\mathbb{F})$ and $\mathcal{Q}_3(\mathbb{F})$ as restrictions of actions $\alpha_1$, $\alpha_2$ and $\alpha_3$ defined on $\mathcal{Q}(\mathbb{F})$. We introduce the following subgroups of $\mathcal{G}(\mathbb{F})$:

$$\mathcal{G}_1(\mathbb{F}) = \{(P, I_m, I_p, 0, 0, 0) \mid P \in Gl_n(\mathbb{F})\}$$

$$\mathcal{G}_2(\mathbb{F}) = \{(P, R, I_p, 0, U, 0) \mid P \in Gl_n(\mathbb{F}), R \in Gl_m(\mathbb{F}), U \in M_{m\times n}(\mathbb{F})\}$$
$$\mathcal{G}_3(\mathbb{F}) = \{(P, I_m, S, T, 0, 0) \mid P \in Gl_n(\mathbb{F}), S \in Gl_p(\mathbb{F}), T \in M_{n\times p}(\mathbb{F})\}$$

It is easy to check the following statement.

**Lemma 1.** *$\mathcal{G}_1(\mathbb{F})$, $\mathcal{G}_2(\mathbb{F})$ and $\mathcal{G}_3(\mathbb{F})$ are closed subgroups of $\mathcal{G}(\mathbb{F})$.*

**Proof.** If $g_1 = (P_1, I_m, I_p, 0, 0, 0)$ and $g_2 = (P_2, I_m, I_p, 0, 0, 0)$ are two elements in $\mathcal{G}_1(\mathbb{F})$, then $g_1 g_2^{-1} = (P_2^{-1}P_1, I_m, I_p, 0, 0, 0)$ is an element in $\mathcal{G}_1(\mathbb{F})$.

If $g_1 = (P_1, R_1, I_p, 0, U_1, 0)$ and $g_2 = (P_2, R_2, I_p, 0, U_2, 0)$ are two elements in $\mathcal{G}_2(\mathbb{F})$, then $g_1 g_2^{-1} = (P_2^{-1}P_1, R_2^{-1}R_1, I_p, 0, -R_2^{-1}U_2P_2^{-1}P_1 + R_2^{-1}U_1, 0)$ is an element in $\mathcal{G}_2(\mathbb{F})$.

If $g_1 = (P_1, I_m, S_1, T_1, 0, 0)$ and $g_2 = (P_2, I_m, S, T_2, 0, 0)$ are two elements in $\mathcal{G}_3(\mathbb{F})$, then $g_1 g_2^{-1} = (P_2^{-1}P_1, I_m, S_1S_2^{-1}, -P_1^{-1}P_2T_2S_2^{-1} + T_1S_2, 0, 0)$ is an element in $\mathcal{G}_3(\mathbb{F})$. $\diamond$

**Remark.** Besides, $\mathcal{G}_1(\mathbb{F})$ is a closed subgroup of $\mathcal{G}_2(\mathbb{F})$ and a closed subgroup of $\mathcal{G}_3(\mathbb{F})$.

We can consider the actions $\alpha_1$, $\alpha_2$, $\alpha_3$ defined as follows.

$$\begin{aligned} \alpha_1 : \mathcal{G}_1(\mathbb{F}) \times \mathcal{Q}(\mathbb{F}) &\longrightarrow \mathcal{Q}(\mathbb{F}) \\ ((P, I_m, I_p, 0, 0, 0), (E, A, B, C)) &\longrightarrow (P^{-1}EP, P^{-1}AP, P^{-1}B, CP) \end{aligned}$$

$$\begin{aligned} \alpha_2 : \mathcal{G}_2(\mathbb{F}) \times \mathcal{Q}(\mathbb{F}) &\longrightarrow \mathcal{Q}(\mathbb{F}) \\ ((P, R, I_p, 0, U, 0), (E, A, B, C)) &\longrightarrow (P^{-1}EP, P^{-1}AP + P^{-1}BU, P^{-1}BR, CP) \end{aligned}$$

$$\begin{aligned} \alpha_3 : \mathcal{G}_3(\mathbb{F}) \times \mathcal{Q}(\mathbb{F}) &\longrightarrow \mathcal{Q}(\mathbb{F}) \\ ((P, I_m, S, T, 0, 0), (E, A, B, C)) &\longrightarrow (P^{-1}EP, P^{-1}AP + TCP, P^{-1}B, SCP) \end{aligned}$$

**Lemma 2.** *The vector subspaces $\mathcal{Q}_1(\mathbb{F})$, $\mathcal{Q}_2(\mathbb{F})$ and $\mathcal{Q}_3(\mathbb{F})$ are invariant under the actions $\alpha_1$, $\alpha_2$ and $\alpha_3$.*

**Proof.** Let us check this statement.

For any $M_1 = (0, A, B, C) \in \mathcal{Q}_1(\mathbb{F})$ and for any $g_1 = (P, I_m, I_p, 0, 0, 0) \in \mathcal{G}_1(\mathbb{F})$, $g_2 = (P, R, I_p, 0, U, 0) \in \mathcal{G}_2(\mathbb{F})$, $g_3 = (P, I_m, S, T, 0, 0) \in \mathcal{G}_3(\mathbb{F})$,

$$\alpha_1(g_1, M_1) = (0, P^{-1}AP, P^{-1}B, CP) \in \mathcal{Q}_1(\mathbb{F})$$

$$\alpha_2(g_2, M_1) = (0, P^{-1}AP + P^{-1}BU, P^{-1}BR, CP) \in \mathcal{Q}_1(\mathbb{F})$$
$$\alpha_3(g_3, M_1) = (0, P^{-1}AP + TCP, P^{-1}B, SCP) \in \mathcal{Q}_1(\mathbb{F})$$

For any $M_2 = (0, A, B, 0) \in \mathcal{Q}_2(\mathbb{F})$ and for any $g_1 = (P, I_m, I_p, 0, 0, 0) \in \mathcal{G}_1(\mathbb{F})$, $g_2 = (P, R, I_p, 0, U, 0) \in \mathcal{G}_2(\mathbb{F})$, $g_3 = (P, I_m, S, T, 0, 0) \in \mathcal{G}_3(\mathbb{F})$,

$$\alpha_1(g_1, M_2) = (0, P^{-1}AP, P^{-1}B, 0) \in \mathcal{Q}_2(\mathbb{F})$$

$$\alpha_2(g_2, M_2) = (0, P^{-1}AP + P^{-1}BU, P^{-1}BR, 0) \in \mathcal{Q}_2(\mathbb{F})$$
$$\alpha_3(g_3, M_2) = (0, P^{-1}AP, P^{-1}B, 0) \in \mathcal{Q}_2(\mathbb{F})$$

For any $M_3 = (0, A, 0, C) \in \mathcal{Q}_3(\mathbb{F})$ and for any $g_1 = (P, I_m, I_p, 0, 0, 0) \in \mathcal{G}_1(\mathbb{F})$, $g_2 = (P, R, I_p, 0, U, 0) \in \mathcal{G}_2(\mathbb{F})$, $g_3 = (P, I_m, S, T, 0, 0) \in \mathcal{G}_3(\mathbb{F})$,

$$\alpha_1(g_1, M_3) = (0, P^{-1}AP, 0, CP) \in \mathcal{Q}_3(\mathbb{F})$$

$$\alpha_2(g_2, M_3) = (0, P^{-1}AP, 0, CP) \in \mathcal{Q}_3(\mathbb{F})$$

$$\alpha_3(g_3, M_3) = (0, P^{-1}AP + TCP, 0, SCP) \in \mathcal{Q}_3(\mathbb{F}) \qquad \diamond$$

For any quadruple $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$, we will denote by $\mathcal{O}_1(E, A, B, C)$, $\mathcal{O}_2(E, A, B, C)$ and $\mathcal{O}_3(E, A, B, C)$ the orbits of this quadruple under the actions $\alpha_1$, $\alpha_2$ and $\alpha_3$.

The equivalence class of the quadruple $(I_n, A, B, C) \in \mathcal{V}_1(\mathbb{F})$ in $\mathcal{V}_1(\mathbb{F})$ under similarity is: $\mathcal{O}_1(I_n, A, B, C) = (I_n, 0, 0, 0) + \mathcal{O}_1(0, A, B, C)$. The equivalence class of the quadruple $(I_n, A, B, 0) \in \mathcal{V}_2(\mathbb{F})$ in $\mathcal{V}_2(\mathbb{F})$ under block-similarity is: $\mathcal{O}_2(I_n, A, B, C) = (I_n, 0, 0, 0) + \mathcal{O}_2(0, A, B, 0)$. The equivalence class of the quadruple $(I_n, A, 0, C) \in \mathcal{V}_3(\mathbb{F})$ in $\mathcal{V}_3(\mathbb{F})$under left block-similarity is: $\mathcal{O}_3(I_n, A, B, C) = (I_n, 0, 0, 0) + \mathcal{O}_3(0, A, 0, C)$.

## 3. Geometrical study of equivalence classes

Let us denote, as usual, by $T_{(E,A,B,C)}\mathcal{O}(E, A, B, C)$ the tangent space to the orbit of the quadruple $(E, A, B, C)$ at $(E, A, B, C)$ under the Lie group action $\alpha$. In a similar way, we denote by $T_{(E,A,B,C)}\mathcal{O}_1(E, A, B, C)$, $T_{(E,A,B,C)}\mathcal{O}_2(E, A, B, C)$ and $T_{(E,A,B,C)}\mathcal{O}_3(E, A, B, C)$ the tangent spaces to the orbits of this quadruple at $(E, A, B, C)$ under the Lie group actions $\alpha_1$, $\alpha_2$ and $\alpha_3$. Then the following characterization of these vector spaces can be given.

**Proposition 1.**

*(a) Let $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$. Then*

$$\begin{aligned} T_{(E,A,B,C)}\mathcal{O}(E, A, B, C) &= \{(EP - PE + BV, AP - PA - BU + TC, BR - PB, CP + SC) \mid \\ &\qquad P \in M_n(\mathbb{F}), R \in M_m(\mathbb{F}), S \in M_p(\mathbb{F}), T \in M_{n\times p}(\mathbb{F}), U, V \in M_{m\times n}(\mathbb{F})\}; \end{aligned}$$

*(b) if $(0, A, B, C) \in \mathcal{Q}_1(\mathbb{F})$,*

$$\begin{aligned} T_{(0,A,B,C)}\mathcal{O}_1(0, A, B, C) &= \{(0, AP - PA, -PB, CP) \mid P \in M_n(\mathbb{F})\}; \\ T_{(0,A,B,C)}\mathcal{O}_2(0, A, B, C) &= \{(0, AP - PA - BU, BR - PB, CP) \mid P \in M_n(\mathbb{F}), R \in M_m(\mathbb{F}), U \in M_{m\times n}(\mathbb{F})\}; \\ T_{(0,A,B,C)}\mathcal{O}_3(0, A, B, C) &= \{(0, AP - PA + TC, -PB, CP + SC) \mid P \in M_n(\mathbb{F}), S \in M_p(\mathbb{F}), T \in M_{n\times p}(\mathbb{F})\}; \end{aligned}$$

*(c) if $(0, A, B, 0) \in \mathcal{Q}_2(\mathbb{F})$,*

$$\begin{aligned} T_{(0,A,B,0)}\mathcal{O}_1(0, A, B, 0) &= \{(0, AP - PA, -PB, 0) \mid P \in M_n(\mathbb{F})\}; \\ T_{(0,A,B,0)}\mathcal{O}_2(0, A, B, 0) &= \{(0, AP - PA - BU, BR - PB, 0) \mid P \in M_n(\mathbb{F}), R \in M_m(\mathbb{F}), U \in M_{m\times n}(\mathbb{F})\}; \\ T_{(0,A,B,0)}\mathcal{O}_3(0, A, B, 0) &= \{(0, AP - PA, -PB, 0) \mid P \in M_n(\mathbb{F})\}; \end{aligned}$$

*(d) if* $(0,A,0,C)\in \mathcal{Q}_3(\mathbb{F})$,

$$\begin{aligned}
T_{(0,A,0,C)}\mathcal{O}_1(0,A,0,C) &= \{(0,AP-PA,0,CP)\mid P\in M_n(\mathbb{F})\};\\
T_{(0,A,0,C)}\mathcal{O}_2(0,A,0,C) &= \{(0,AP-PA,0,CP)\mid P\in M_n(\mathbb{F})\};\\
T_{(0,A,0,C)}\mathcal{O}_3(0,A,0,C) &= \{(0,AP-PA+TC,0,CP+SC)\mid P\in M_n(\mathbb{F}), S\in M_p(\mathbb{F}), T\in M_{n\times p}(\mathbb{F})\}.
\end{aligned}$$

**Proof:** Considering the expansions of $\alpha(I+\varepsilon g,(E,A,B,C))$, $g\in\mathcal{G}$, $\alpha_1(I+\varepsilon g_1,(0,A,B,C))$, $g_1\in\mathcal{G}_1$, $\alpha_2(I+\varepsilon g_2,(0,A,B,0))$, $g_2\in\mathcal{G}_2$, and $\alpha_3(I+\varepsilon g_3,(0,A,0,C))$, $g_3\in\mathcal{G}_3$, up to first order term in $\varepsilon$, it is not difficult to check that the statement holds. $\diamond$

As a consequence, it is immediate to prove the following Corollary.

**Corollary 1.**

*(a)* $T_{(I_n,A,B,C)}\mathcal{O}_1(I_n,A,B,C)=(I_n,0,0,0)+T_{(0,A,B,C)}\mathcal{O}_1(0,A,B,C)$;

*(b)* $T_{(I_n,A,B,0)}\mathcal{O}_2(I_n,A,B,0)=(I_n,0,0,0)+T_{(0,A,B,0)}\mathcal{O}_2(0,A,B,C)$;

*(c)* $T_{(I_n,A,0,C)}\mathcal{O}_3(I_n,A,0,C)=(I_n,0,0,0)+T_{(0,A,0,C)}\mathcal{O}_3(0,A,0,C)$.

**Remark.** Note that $T_{(0,A,B,C)}\mathcal{O}_1(0,A,B,C)\neq T_{(0,A,B,C)}\mathcal{O}(0,A,B,C)$, $T_{(0,A,B,0)}\mathcal{O}_2(0,A,B,0)\neq T_{(0,A,B,0)}\mathcal{O}(0,A,B,0)$ but $T_{(0,A,0,C)}\mathcal{O}_3(0,A,0,C)=T_{(0,A,0,C)}\mathcal{O}(0,A,0,C)$.

Let us consider the following matrices:

$$T(E,A,B,C)=\begin{pmatrix} I_n\otimes E-E^t\otimes I_n & 0 & 0 & 0 & 0 & I_n\otimes B\\ I_n\otimes A-A^t\otimes I_n & 0 & 0 & C^t\otimes I_n & I_n\otimes B & 0\\ -B^t\otimes I_n & I_m\otimes B & 0 & 0 & 0 & 0\\ I_n\otimes C & 0 & C^t\otimes I_p & 0 & 0 & 0\end{pmatrix}$$

$$T_1(0,A,B,C)=\begin{pmatrix} 0\\ I_n\otimes A-A^t\otimes I_n\\ -B^t\otimes I_n\\ I_n\otimes C\end{pmatrix}$$

$$T_2(0,A,B,0)=\begin{pmatrix} 0 & 0 & 0\\ I_n\otimes A-A^t\otimes I_n & 0 & I_n\otimes B\\ -B^t\otimes I_n & I_m\otimes B & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$T_3(0,A,0,C)=\begin{pmatrix} 0 & 0 & 0\\ I_n\otimes A-A^t\otimes I_n & 0 & C^t\otimes I_n\\ 0 & 0 & 0\\ I_n\otimes C & C^t\otimes I_p & 0\end{pmatrix}$$

These matrices allow us to present a characterization of the tangent spaces, which will be useful in next Sections.

**Proposition 2.**

*(a) Given any quadruple* $(E, A, B, C)$ *in* $\mathcal{Q}(\mathbb{F})$, $T_{(E,A,B,C)}\mathcal{O}(E, A, B, C) = \operatorname{Im} T(E, A, B, C)$.

*(b) Given any quadruple* $(0, A, B, C)$ *in* $\mathcal{Q}_1(\mathbb{F})$, $T_{(0,A,B,C)}\mathcal{O}_1(0, A, B, C) = \operatorname{Im} T_1(0, A, B, C)$.

*(c) Given any quadruple* $(0, A, B, 0)$ *in* $\mathcal{Q}_2(\mathbb{F})$, $T_{(0,A,B,0)}\mathcal{O}_2(0, A, B, 0) = \operatorname{Im} T_2(0, A, B, 0)$.

*(d) Given any quadruple* $(0, A, 0, C)$ *in* $\mathcal{Q}_3(\mathbb{F})$, $T_{(0,A,0,C)}\mathcal{O}_3(0, A, 0, C) = \operatorname{Im} T_3(0, A, 0, C)$.

**Proof:** The proof is based on the properties of the vec operator (see [LT85] for its definition and properties) and its relationship with the Kronecker product. We will explicitly given the proof of part (a), the other parts can be handled analogously.

According to Proposition 1, we know that $(E', A', B', C') \in T_{(E,A,B,C)}\mathcal{O}(E, A, B, C)$ if, and only if, there exist $P \in M_n(\mathbb{F})$, $R \in M_m(\mathbb{F})$, $S \in M_p(\mathbb{F})$, $T \in M_{n\times p}(\mathbb{F})$, $U, V \in M_{m\times n}(\mathbb{F})$ such that

$$E' = EP - PE + BV,\ A' = AP - PA - BU + TC,\ B' = BR - PB,\ C' = CP + SC$$

Equivalently,

$$\begin{aligned}
vec(E') &= (I_n \otimes E - E^t \otimes I_n)vec(P) + (I_n \otimes B)vec(V)\\
vec(A') &= (-A^t \otimes I_n + I_n \otimes A)vec(P) + (C^t \otimes I_n)vec(T) + (I_n \otimes B)vec(U)\\
vec(B') &= (-B^t \otimes I_n)vec(P) + (I_m \otimes B)vec(R)\\
vec(C') &= (I_n \otimes C)vec(P) + (C^t \otimes I_p)vec(S)
\end{aligned}$$

or, with a matritial notation,

$$\begin{pmatrix} vec(E') \\ vec(A') \\ vec(B') \\ vec(C') \end{pmatrix} = T(E, A, B, C) \begin{pmatrix} vec(P) \\ vec(R) \\ vec(S) \\ vec(T) \\ vec(U) \\ vec(V) \end{pmatrix} \qquad \diamond$$

## 4. Distances in $\mathcal{Q}(\mathbb{F})$

The distances we will deal with are those deduced from the Frobenius norm and the 2-norm. We briefly recall their definition.

Given a matrix $M = (m^i_j)_{1\le i\le m, 1\le j\le n}$ with $m$ rows and $n$ columns, its Frobenius norm is defined as

$$\|M\|_F = \sqrt{\sum_{1\le i\le m, 1\le j\le n} m^i_j \overline{m}^i_j}$$

and its 2-norm is defined as the largest singular value of $M$. We will denote it by $\sigma_1(M)$. If $\operatorname{rank} M = r$, then $\sigma_r(M)$ is the smallest non-zero singular value of $M$: $\sigma_r(M) > \sigma_{r+1}(M) = \ldots = \sigma_m(M) = 0$.

The norms above lead to the natural definition of the Frobenius norm and the 2-norm of quadruples in $\mathcal{Q}(\mathbb{F})$, and the corresponding definition of the Frobenius distance and the 2-distance in $\mathcal{Q}(\mathbb{F})$.

**Definition 5.** Given a quadruple $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$ we define its Frobenius norm as

$$\|(E, A, B, C)\|_F = \sqrt{\|E\|_F^2 + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2}$$

and thus the Frobenius distance between the quadruples $(E_1, A_1, B_1, C_1)$ and $(E_2, A_2, B_2, C_2)$ is

$$d_F((E_1, A_1, B_1, C_1), (E_2, A_2, B_2, C_2)) = \|(E_1 - E_2, A_1 - A_2, B_1 - B_2, C_1 - C_2)\|_F$$

**Definition 6.** The 2-norm of the quadruple $(E, A, B, C)$ is defined as the 2-norm of the matrix $\begin{pmatrix} E & A & B \\ 0 & C & 0 \end{pmatrix}$. And the 2-distance between the quadruples $(E_1, A_1, B_1, C_1)$ and $(E_2, A_2, B_2, C_2)$ is

$$d_2((E_1, A_1, B_1, C_1), (E_2, A_2, B_2, C_2)) = \|(E_1 - E_2, A_1 - A_2, B_1 - B_2, C_1 - C_2)\|_2$$

Finally, we can define the distance between a quadruple satisfying a property and the nearest one not satisfying it.

**Definition 7.** Given any norm (for example those above), the distance between the quadruple $(E, A, B, C)$ which satisfies a property and the nearest quadruple non-satisfying it is considered to be:

$$\inf \|(E + \delta E, A + \delta A, B + \delta B, C + \delta C)\|$$

where $(\delta E, \delta A, \delta B, \delta C)$ is a quadruple such that $(E + \delta E, A + \delta A, B + \delta B, C + \delta C)$ does not satisfy the given property.

## 5. Controllability and observability properties

Controllability and observability are two qualitative properties of linear dynamical systems. They are very important in the study of control and filtering problems.

Let us consider a quadruple of matrices $(I_n, A, B, C) \in \mathcal{Q}_1(\mathbb{F})$ defining a linear multivariable time-invariant dynamical system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with $A$ a square matrix and $B, C$ rectangular matrices.

**Definition 8.** We say that the state equation $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable when the transfer of any state to any other state can be achieved in a non-zero time interval.

The controllability matrix of the pair $(A, B)$ is defined as

$$\mathbf{C}(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

It is well-known that the system is controllable if, and only if, the matrix $\mathbf{C}(A, B)$ has full rank.

**Definition 9.** The system above is said to be observable when the determination of the initial state can be achieved in any non-zero time interval.

The concept of observability is dual to the concept of controllability. Hence there is a similar criterion giving a necessary and sufficient condition for a system to be observable.

The observability matrix of the pair $(C, A)$ is defined as

$$\mathbf{O}(C, A) = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{pmatrix}$$

It is well-known that the system is observable if, and only if, the matrix $\mathbf{O}(C, A)$ has full rank.

One also knows that a system is controllable and observable if, and only if, the rank of the controllability-observability matrix,

$$\mathbf{CO}(A, B, C) = \mathbf{O}(C, A) \cdot \mathbf{C}(A, B) = \begin{pmatrix} CB & CAB & \dots & CA^{n-1}B \\ CAB & CA^2B & \dots & CA^nB \\ \dots & \dots & & \dots \\ CA^{n-1}B & CA^nB & \dots & CA^{2(n-1)}B \end{pmatrix}$$

has full rank (this follows from Sylvester's inequality; see [Ga77] for details).

## 6. Structural stability

Many mathematical objects are known only approximately. In the case of a topological space with an equivalence relation defined in it, an element such that one can find an open neighbourhood containing only elements equivalent to it is called a "structurally stable" element (see [Wi80]). That is to say, a "structurally stable" element is an element whose behaviour does not change when suffering small perturbations.

The concept of structural stability was first introduced by Andronov and Pontryagin ([AP37]) in the qualitative theory of dynamical systems and has been widely studied by many authors (see ([Ar71], [Ta81], [Wi80], etc.). We will consider the concept of structural stability as appears in [Wi80].

**Definition 10.** ([Wi80], p. 313). Let $X$ be a topological space where an equivalence relation is defined. An element $x \in X$ is said to be structurally stable if, and only if, there exists an open neighbourhood $\mathcal{U}$ of it in $X$ such that for all $x' \in \mathcal{U}$, $x'$ is equivalent to $x$.

**Remark.** In the case where the topological space $X$ is a differentiable or complex manifold and the equivalence relation is that induced by the action of a Lie group, giving rise to orbits which are (differentiable or complex) submanifolds, then it is a straightforward consequence of the definition above that the following statements are equivalent:

1. $x$ is structurally stable;
2. the orbit of $x$, $\mathcal{O}(x)$, is an open manifold;
3. $\dim \mathcal{O}(x) = \dim X$.
4. $\dim T_x\mathcal{O}(x) = \dim X$.

Structurally stable elements have been studied in the case of the linear group acting on the space of square matrices (see [Ar71]). The characterization of structurally stable pairs of matrices, under block-similarity, in terms of their discrete invariants is presented in [Ga94]. Also different characterizations of structurally stable quadruples of matrices $(E, A, B, C)$, with respect to an equivalence relation, generalizing feedback equivalence, are given in [GM00].

## 7. Bounding the distance from structurally stable quadruples to structurally unstable ones

The geometrical study of equivalence classes made in Section §3 yields the following characterization of quadruples which are structurally stable under the equivalence relations considered in Section §2.

**Proposition 3.**

*(a) A quadruple* $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$ *is structurally stable with respect to feedback equivalence (see Definition 1) if, and only if,* $\operatorname{rank} T(E, A, B, C) = 2n^2 + mn + np$.

*(b) There are no structurally stable quadruples* $(I_n, A, B, C)$ *in* $\mathcal{V}_1(\mathbb{F})$ *with respect to similarity (see Definition 2).*

*(c) A quadruple* $(I_n, A, B, 0) \in \mathcal{V}_2(\mathbb{F})$ *is structurally stable with respect to block-similarity (see Definition 3) if, and only if,* $\operatorname{rank} T_2(0, A, B, 0) = n^2 + mn$.

*(d) A quadruple* $(I_n, A, 0, C) \in \mathcal{V}_3(\mathbb{F})$ *is structurally stable with respect to left block-similarity (see Definition 4) if, and only if,* $\operatorname{rank} T_3(0, A, 0, C) = n^2 + np$.

**Proof.** The characterizations are a straightforward consequence of the definition of structurally stable element.

Part (b) follows from the fact that a quadruple $(I_n, A, B, C) \in \mathcal{V}_1(\mathbb{F})$ would be structurally stable with respect to similarity if, and only if, $\operatorname{rank} T_1(0, A, B, C) = n^2 + mn + np$ and it is obvious that $\operatorname{rank} T_1(0, A, B, C) \leq n^2$. $\diamond$

Our goal is to obtain a bound for the value of the radius of a ball which is a neighbourhood of a structurally stable element, containing only elements which are also structurally stable.

Edelman, Elmroth and Kågström (see [EEK97]), as well as other authors, have studied linear systems which can be represented in the form $\dot{x}(t) = Ax(t) + Bu(t)$ with multiple inputs and outputs (no derivative feedback or output injection transformations being considered) by associating the matrix pencil $(A\ B) - \lambda(I_n\ 0)$ to the pair $(A, B)$ and considering the tangent space to this pencil. Then the equivalence relation is equivalent to the strict equivalence of matrix pencils. But a perturbed matrix pencil does not necessarily represent a pair of matrices. For example, consider the quadruple $(I_n, A, B, 0)$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The matrix pencil

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a perturbed pencil but it is not associated to any quadruple of matrices of the form $(I_n, A, B, 0)$. We will only consider perturbed quadruples, being the bound thus obtained an improvement for a safety neighbourhod (see Examples 1 and 2).

The starting point to find a bound is the relationship between the Frobenius norm of a quadruple and the matrices associated to it in Section §3, given in terms of a constant which depends only on the order of the matrices of the quadruple. More concretely, we have the following result.

**Proposition 4.**

*(a) For all* $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$, $\|T(E, A, B, C)\|_F \leq \sqrt{3n + m + p}\ \|(E, A, B, C)\|_F$.

*(b) For all* $(0, A, B, 0) \in \mathcal{Q}_2(\mathbb{F})$, $\|T_2(0, A, B, 0)\|_F \leq \sqrt{2n + m}\ \|(0, A, B, 0)\|_F$.

*(c) For all* $(0, A, 0, C) \in \mathcal{Q}_3(\mathbb{F})$, $\|T_3(0, A, 0, C)\|_F \leq \sqrt{2n + p}\ \|(0, A, 0, C)\|_F$.

**Proof:** (a) By direct calculation the following equality can be checked:

$$\|T(E, A, B, C)\|_F^2 \leq 2n\|E\|_F^2 + 2n\|A\|_F^2 + (3n + m)\|B\|_F^2 + (2n + p)\|C\|_F^2$$

Thus,

$$\begin{aligned} \|T(E, A, B, C)\|_F^2 &\leq (3n + m + p)\ (\|E\|_F^2 + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2) \\ &= (3n + m + p)\ \|(E, A, B, C)\|_F^2. \end{aligned}$$

(b) Analogously as in (a), the statement follows from the inequality:

$$\|T_2(0, A, B, 0)\|_F^2 \leq 2n\|A\|_F^2 + m\|B\|_F^2 \leq (2n + m)(\|A\|_F^2 + \|B\|_F^2)$$

(c) Analogously as in (a), the statement follows from the inequality:

$$\|T_2(0, A, 0, C)\|_F^2 \leq 2n\|A\|_F^2 + p\|C\|_F^2 \leq (2n + p)(\|A\|_F^2 + \|C\|_F^2) \qquad \diamond$$

Let us assume $(E, A, B, C)$ is a structurally stable quadruple of matrices with respect to one of the equivalence relations defined in Section §2. A bound for the distance from this quadruple to the nearest structurally unstable one, $(E + \delta E, A + \delta A, B + \delta B, C + \delta C)$, with respect to feedback-similarity, block-similarity or left block-similarity, is given in the next Theorem.

**Theorem 1.**

*(a) For a given structurally stable quadruple of matrices $(E, A, B, C) \in \mathcal{Q}(\mathbb{F})$, with respect to feedback equivalence, a lower bound for the distance to the nearest structurally unstable quadruple is given by*

$$\|(\delta E, \delta A, \delta B, \delta C)\|_F \geq \frac{1}{\sqrt{3n+m+p}}\ \sigma_{n^2+mn+np}(T(E, A, B, C))$$

*where $\sigma_{n^2+mn+np}(T(E, A, B, C))$ denotes the smallest non-zero singular value of $T(E, A, B, C)$.*

*(b) For a given structurally stable quadruple of matrices $(I_n, A, B, 0) \in \mathcal{V}_2(\mathbb{F})$, with respect to block-similarity, a lower bound for the distance to the nearest structurally unstable quadruple in $\mathcal{V}_2(\mathbb{F})$ is given by*

$$\|(0, \delta A, \delta B, 0)\|_F \geq \frac{1}{\sqrt{2n+m}}\ \sigma_{n^2+mn}(T_2(0, A, B, 0))$$

*where $\sigma_{n^2+mn}(T_2(0, A, B, 0))$ denotes the smallest non-zero singular value of $T_2(0, A, B, 0)$.*

*(c) For a given structurally stable quadruple of matrices $(I_n, A, 0, C) \in \mathcal{V}_3(\mathbb{F})$, with respect to left block-similarity, a lower bound for the distance to the nearest structurally unstable quadruple in $\mathcal{V}_3(\mathbb{F})$ is given by*

$$\|(0, \delta A, 0, \delta C)\|_F \geq \frac{1}{\sqrt{2n+p}}\ \sigma_{n^2+np}(T_3(0, A, 0, C))$$

*where $\sigma_{n^2+np}(T_3(0, A, 0, C))$ denotes the smallest non-zero singular value of $T_3(0, A, 0, C)$.*

**Proof:** (a) We know that $\operatorname{rk} T(E, A, B, C) = 2n^2 + mn + np$ and that if $(E+\delta E, A+\delta A, B+\delta B, C+\delta C)$ is not structurally stable, $\operatorname{rk} T(E+\delta E, A+\delta A, B+\delta B, C+\delta C) \leq 2n^2+mn+np-1$.

The Eckart-Young and Minkowski Theorem states that the smallest perturbation in the Frobenius norm that reduces the rank of a matrix $M$ with $\operatorname{rank} M = r$ from $r$ to $r-1$ is $\sigma_r(M)$, the smallest non-zero singular value of $M$. Thefore the norm of the perturbation of the $T$-matrix, $\|(\delta E, \delta A, \delta B, \delta C)\|_F$ must be at least $\sigma_{2n^2+mn+np}(T(E, A, B, C))$. The only fact which needs to be noted is that

$$T(E+\delta E, A+\delta A, B+\delta B, C+\delta C) = T(E, A, B, C) + T(\delta E, \delta A, \delta B, \delta C)$$

which yields

$$\|T(E+\delta E, A+\delta A, B+\delta B, C+\delta C)\|_F \leq \|T(E, A, B, C)\|_F + \|T(\delta E, \delta A, \delta B, \delta C)\|_F.$$

Hence, a bound for the distance from $(E, A, B, C)$ to the nearest structurally unstable quadruple, taking into account Proposition 4, is:

$$\|(\delta E, \delta A, \delta B, \delta C)\|_F \ \geq \tfrac{1}{\sqrt{3n+m+p}}\ \|T(\delta E, \delta A, \delta B, \delta C)\|_F \geq \tfrac{1}{\sqrt{3n+m+p}}\ \sigma_{2n^2+mn+np}(T(E, A, B, C))$$

Parts (b) and (c) can be proved analogously. $\diamond$

## 8. Bounding the distance from a controllable and observable system to an uncontrollable and/or unobservable one

It is well-known that the set of controllable and observable triples of matrices is an open dense set in the space of all triples of matrices $(A, B, C)$, which can be identified with $\mathcal{V}_1(\mathbb{F})$. Also the set of controllable pairs of matrices is an open dense set in the space of all pairs of matrices $(A, B)$, which can be identified with $\mathcal{V}_2(\mathbb{F})$ and the set of observable pairs of matrices is an open dense set in the space of all pairs of matrices $(A, C)$, which can be identified with $\mathcal{V}_3(\mathbb{F})$.

For each controllable and observable triple there exists an open neighbourhood of the triple such that all the triples in it are controllable and observable. Then it makes sense to consider the distance to the nearest uncontrollable and/or unobservable triple and to deduce a safety neighbourhood.

Eising in [Ei84] measured the distance between a controllable pair of matrices and the nearest uncontrollable pair, as

$$d_{\mathbb{C}}(A, B) = \min_{k\in\mathbb{C}} \sigma_n(kI_n - A, B)$$

where $\sigma_n(kI_n - A, B)$ is the smallest singular value of $(kI_n - A, B)$. The computation of this bound is an involved process and the analogous result is not true in the case $\mathbb{F} = \mathbb{R}$. Consider, for example, the controllable pair of matrices $(A, B)$ with

$$A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The smallest non-zero singular value of $(kI_2 - A, B)$, for $k \in \mathbb{C}$ is 0.6959705454. And the smallest non-zero singular value of $(kI_2 - A, B)$, for $k \in \mathbb{R}$ is 2. The pair $(A + \delta A, B + \delta B)$, with

$$\delta A = 0, \qquad \delta B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

is obviously uncontrollable and

$$d_{2,\mathbb{R}}((A, B), (A + \delta A, B + \delta B)) = \|(\delta A, \delta B)\|_2 = 1$$

We present in this Section upper bounds which are given in terms of the singular values of the controllability, observability and controllability-observability matrices, hence they are easily computable.

Let us assume $\mathbb{F} = \mathbb{R}$.

Let us denote by $\begin{bmatrix}\Sigma^{\mathbf{c}} & 0\end{bmatrix}$, $\begin{bmatrix}\Sigma^{\mathbf{o}} \\ 0\end{bmatrix}$, $\begin{bmatrix}\Sigma^{\mathbf{co}} & 0 \\ 0 & 0\end{bmatrix}$ the singular value decomposition of the controllability matrix of the pair $(A, B)$, the observability matrix of the pair $(C, A)$ and the controllability-observability matrix of the triple $(A, B, C)$. There exist orthogonal matrices $X_{\mathbf{c}}, Y_{\mathbf{c}}$, $X_{\mathbf{o}}, Y_{\mathbf{o}}$ and $X_{\mathbf{co}}, Y_{\mathbf{co}}$ such that

$$\mathbf{C}(A, B) = X_{\mathbf{c}}^t \begin{bmatrix}\Sigma^{\mathbf{c}} & 0\end{bmatrix} Y_{\mathbf{c}}$$

$$\mathbf{O}(C,A) = X_{\mathbf{O}}^t \left[ \begin{array}{c} \Sigma^{\mathbf{O}} \\ 0 \end{array} \right] Y_{\mathbf{O}}$$

$$\mathbf{CO}(A,B,C) = X_{\mathbf{CO}}^t \left[ \begin{array}{cc} \Sigma^{\mathbf{CO}} & 0 \\ 0 & 0 \end{array} \right] Y_{\mathbf{CO}}$$

We denote by

$$\sigma_1^{\mathbf{C}} \geq \sigma_2^{\mathbf{C}} \geq \ldots \geq \sigma_n^{\mathbf{C}} > 0$$
$$\sigma_1^{\mathbf{O}} \geq \sigma_2^{\mathbf{O}} \geq \ldots \geq \sigma_n^{\mathbf{O}} > 0$$
$$\sigma_1^{\mathbf{CO}} \geq \sigma_2^{\mathbf{CO}} \geq \ldots \geq \sigma_n^{\mathbf{CO}} > 0$$

the singular values of $\mathbf{C}(A,B)$, $\mathbf{O}(C,A)$ and $\mathbf{CO}(A,B,C)$, respectively.

In [BL86] a bound for the distance from a controllable pair to the nearest uncontrollable one is given, after proving the following Lemma.

**Lemma 4.** ([BL86], p. 250) *For a given quadruple of matrices $(I_n, A, B, 0)$, with $(A,B)$ controllable, for all $i \in \{1,\ldots,n-1\}$ there exists an orthogonal matrix $P$ such that*

$$A' = PAP^t = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B' = PB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

*where $A_1 \in M_i(\mathbb{R}), B_1 \in M_{i\times m}(\mathbb{R})$, with*

$$\|A_3\|_2 \leq \|A_c\|_2 \frac{\sigma_{i+1}^{\mathbf{C}}}{\sigma_i^{\mathbf{C}}}, \quad \|B_2\|_2 \leq \sigma_{i+1}^{\mathbf{C}}.$$

The reasonement in the proof of Theorem 6 in [BL86] provides the following bound for the distance from a controllable quadruple of matrices and the nearest uncontrollable one.

**Theorem 2.** *An upper bound for the distance between the controllable quadruple $(I_n, A, B, 0) \in \mathcal{V}_2(\mathbb{R})$ and the nearest uncontrollable one is given by:*

$$\mu_{2,\mathbb{R}}^{\mathbf{C}}(I_n, A, B, 0) = \min \left\{ \left(1 + \frac{\|A_c\|_2}{\sigma_1^{\mathbf{C}}}\right) \sigma_2^{\mathbf{C}}, \ldots, \left(1 + \frac{\|A_c\|_2}{\sigma_{n-1}^{\mathbf{C}}}\right) \sigma_n^{\mathbf{C}} \right\}$$

*where $\sigma_i^{\mathbf{C}}$ are the singular values of the controllability matrix of the pair $(A,B)$ and $A_c$ is the companion matrix of $A$.*

Observability is the dual concept of controllability. This duality allows to state an analogous result to that in Theorem 2 to bound the distance from an observable quadruple of matrices and the nearest unobservable one.

**Theorem 3.** *An upper bound for the distance between the observable quadruple $(I_n, A, 0, C) \in \mathcal{V}_3(\mathbb{R})$ and the nearest unobservable one is given by:*

$$\mu_{2,\mathbb{R}}^{\mathbf{O}}(I_n, A, 0, C) = \min \left\{ \left(1 + \frac{\|A_c\|_2}{\sigma_1^{\mathbf{O}}}\right) \sigma_2^{\mathbf{O}}, \ldots, \left(1 + \frac{\|A_c\|_2}{\sigma_{n-1}^{\mathbf{O}}}\right) \sigma_n^{\mathbf{O}} \right\}$$

*where $\sigma_i^{\mathbf{o}}$ are the singular values of the observability matrix of the pair $(C, A)$ and $A_c$ is the companion matrix of $A$.*

**Remark.** Let us denote by $d^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, C), d^{\mathbf{c}}_{2,\mathbb{C}}(I_n, A, B, C)$ the 2-distances from the controllable quadruple $(I_n, A, B, C)$ to the nearest uncontrollable quadruple in the real and complex cases, respectively, and by $d^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C), d^{\mathbf{o}}_{2,\mathbb{C}}(I_n, A, B, C)$ the 2-distances from the observable quadruple $(I_n, A, B, C)$ to the nearest unobservable quadruple in the real and complex cases, respectively. Then $d^{\mathbf{c}}_{2,\mathbb{C}}(I_n, A, B, C) \leq d^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, C), d^{\mathbf{o}}_{2,\mathbb{C}}(I_n, A, B, C) \leq d^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C)$ and hence the bounds in Theorems 2, 3 are also bounds in the complex case.

Let $(I_n, A, B, C) \in \mathcal{V}_1(\mathbb{R})$ be a controllable and observable quadruple of matrices. We denote by $d^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C)$ the distance between this quadruple and the nearest uncontrollable and unobservable one. It is obviuos that

$$d^{\mathbf{co}}_{2,\mathbb{F}}(I_n, A, B, C) \geq \max\{d^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, C), d^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C)\}$$

Controllability is not an invariant property under left block-similarity or feedback equivalence, but under block-similarity (hence under similarity). Observability is an invariant property under left block-similarity (hence under similarity) but not under block-similarity or feedback equivalence. Controllability and observability property is not invariant under neither block-similarity, left block-similarity nor feedback equivalence, but under similarity.

In particular, all the quadruples in the orbit of $(I_n, A, B, C)$ under $\alpha_2$ are controllable and $d^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, 0)$ is the distance from $(I_n, A, B, C)$ to the nearest orbit consisting of uncontrollable quadruples of matrices. Analogously, all the quadruples in the orbit of $(I_n, A, B, C)$ under $\alpha_3$ are observable and $d^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C)$ is the distance from $(I_n, A, B, C)$ to the nearest orbit consisting of unobservable quadruples of matrices and all the quadruples in the orbit of $(I_n, A, B, C)$ under $\alpha_1$ are controllable and observable, and $d^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C)$ is a bound for the distance from $(I_n, A, B, C)$ to the nearest orbit consisting of uncontrollable and/or unobservable quadruples of matrices.

$$d^{\mathbf{co}}_{2,\mathbb{F}}(I_n, A, B, C) \geq \max\{d^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, 0), d^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, 0, C)\}$$

Given any $(I_n, A, B, C) \in \mathcal{V}_1(\mathbb{R})$ controllable and observable quadruple of matrices, we will find a bound for $d^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C)$. The method we use is similar to that in [BL86], exploring the singular values of the associated controllability, observability and controllability-observability matrices.

The bound will be derived from the statement in next Lemma, which is similar to Lemma 5 in [BL86].

**Lemma 5.** *Given a quadruple $(I_n, A, B, C)$ which is controllable and observable, for all $i \in \{1, \ldots, n-1\}$, there exists an orthogonal matrix $P$ such that*

$$A' = PAP^t = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B' = PB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C' = CP^t = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

*where* $A_1 \in M_i(\mathbb{R})$*,* $B_1 \in M_{i\times m}(\mathbb{R})$*,* $C_1 \in M_{p\times i}(\mathbb{R})$ *and*

$$\|A_2\|_2 \le \|A_c\|_2 \frac{\sigma^{\mathbf{O}}_{i+1}}{\sigma^{\mathbf{O}}_i}, \quad \|B_1\|_2 \le \min\left\{\frac{\sigma^{\mathbf{CO}}_1}{\sigma^{\mathbf{O}}_i}, \sigma^{\mathbf{C}}_1\right\}, \quad \|C_2\|_2 \le \sigma^{\mathbf{O}}_{i+1}$$

**Proof.** We will make use of the notations at the beginning of this Section.

Let us consider the quadruple $(I_n, A', B', C')$, where $A' = Y_{\mathbf{O}} A Y^t_{\mathbf{O}}$, $B' = Y_{\mathbf{O}} B$, $C' = C Y^t_{\mathbf{O}}$.

Then

$$\mathbf{O}(C,A)B = X^t_{\mathbf{O}} \begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix} Y_{\mathbf{O}} B = X^t_{\mathbf{O}} \begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix} B'$$

On the other hand,

$$\mathbf{O}(C,A)B = \begin{pmatrix} CB \\ CAB \\ \dots \\ CA^{n-1}B \end{pmatrix} = \mathbf{CO}(A,B,C) \begin{pmatrix} I_m \\ 0 \end{pmatrix} = X^t_{\mathbf{CO}} \begin{bmatrix} \Sigma^{\mathbf{CO}} & 0 \\ 0 & 0 \end{bmatrix} Y_{\mathbf{CO}} \begin{pmatrix} I_m \\ 0 \end{pmatrix}$$

thus

$$B' = \begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix}^+ (X^t_{\mathbf{O}})^+ X^t_{\mathbf{CO}} \begin{bmatrix} \Sigma^{\mathbf{CO}} & 0 \\ 0 & 0 \end{bmatrix} Y_{\mathbf{CO}} \begin{pmatrix} I_m \\ 0 \end{pmatrix}$$

where $M^+$ denotes the Moore-Penrose inverse of the matrix $M$. If

$$\begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \\ 0 & 0 \end{pmatrix}$$

with

$$D_1 = \mathrm{diag}\,(\sigma^{\mathbf{O}}_1, \dots, \sigma^{\mathbf{O}}_i), \quad D_2 = \mathrm{diag}\,(\sigma^{\mathbf{O}}_{i+1}, \dots, \sigma^{\mathbf{O}}_n)$$

then

$$\begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix}^+ = \begin{pmatrix} D_1^{-1} & 0 & 0 \\ 0 & D_2^{-1} & 0 \end{pmatrix}$$

We partition into blocks the matrices in the expression above, in the following way:

$$\begin{aligned} B' &= \begin{pmatrix} D_1^{-1} & 0 & 0 \\ 0 & D_2^{-1} & 0 \end{pmatrix} \begin{pmatrix} X_1 & X'_1 \\ X_2 & X'_2 \\ X_3 & X'_3 \end{pmatrix} \begin{bmatrix} \Sigma^{\mathbf{CO}} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y'_1 & Y'_2 & Y'_3 \end{pmatrix} \begin{pmatrix} I_i & 0 \\ 0 & I_{m-i} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} D_1^{-1} X_1 \Sigma^{\mathbf{CO}} Y_1 \\ D_2^{-1} X_3 \Sigma^{\mathbf{CO}} Y_1 \end{pmatrix} \end{aligned}$$

thus obtaining

$$\|B_1\|_2 \le \|D_1^{-1}\|_2 \|X_1\|_2 \|\Sigma^{\mathbf{CO}}\|_2 \|Y_1\|_2 \le (\sigma^{\mathbf{O}}_i)^{-1} \sigma^{\mathbf{CO}}_1$$

Note that $\mathbf{CO}(A,B,C) = \mathbf{O}(C,A) \cdot \mathbf{C}(A,B)$ implies

$$\left(X^t_{\mathbf{O}}\right)^+ X^t_{\mathbf{CO}} \begin{bmatrix} \Sigma^{\mathbf{CO}} & 0 \\ 0 & 0 \end{bmatrix} Y_{\mathbf{CO}} = \begin{bmatrix} \Sigma^{\mathbf{O}} \\ 0 \end{bmatrix} Y_{\mathbf{O}} X^t_{\mathbf{C}} \begin{bmatrix} \Sigma^{\mathbf{C}} & 0 \end{bmatrix} Y_{\mathbf{C}}$$

and then

$$B' = Y_{\mathbf{O}} X_{\mathbf{C}}^t \left[ \Sigma^{\mathbf{C}} \quad 0 \right] Y_{\mathbf{C}} \begin{pmatrix} I_m \\ 0 \end{pmatrix}$$

hence we have also the bound

$$\|B_1\|_2 \leq \|B'\|_2 \leq \sigma_1^{\mathbf{C}}$$

Lemma 5 in [BL86] yields, considering the controllable pair of matrices $(A'^t, C'^t)$:

$$\|A_2\|_2 \leq \|A_c\|_2 \frac{\sigma_{i+1}^{\mathbf{O}}}{\sigma_i^{\mathbf{O}}}, \quad \|C_2\|_2 \leq \sigma_{i+1}^{\mathbf{O}}$$

for all $i \in \{1, \ldots, n-1\}$. $\diamond$

Finally, we can explicitly give an upper bound.

**Theorem 4.** *Let $(I_n, A, B, C)$ be a controllable and observable quadruple of matrices. An upper bound for the distance from this quadruple to the nearest uncontrollable and unobservable quadruple is given by*

$$\mu_{2,\mathbb{R}}^{\mathbf{CO}}(I_n, A, B, C) = \min \left\{ \left( \|A_c\|_2 \sigma_2^{\mathbf{O}} + \min \left\{ \sigma_1^{\mathbf{CO}}, \sigma_1^{\mathbf{C}} \sigma_1^{\mathbf{O}} \right\} \right) \frac{1}{\sigma_1^{\mathbf{O}}} + \sigma_2^{\mathbf{O}}, \ldots, \left( \|A_c\|_2 \sigma_n^{\mathbf{O}} + \min \left\{ \sigma_1^{\mathbf{CO}}, \sigma_1^{\mathbf{C}} \sigma_{n-1}^{\mathbf{O}} \right\} \right) \frac{1}{\sigma_{n-1}^{\mathbf{O}}} + \sigma_n^{\mathbf{O}} \right\}$$

*where $\sigma_i^{\mathbf{O}}$, $\sigma_i^{\mathbf{CO}}$ are the singular values of the observability matrix of the pair $(C, A)$ and of the controllability-observbility of the triple $(A, B, C)$, respectively, and $A_c$ is the companion matrix of $A$.*

**Proof.** We will prove that, for all $i \in \{1, \ldots, n-1\}$,

$$d_{2,\mathbb{R}}^{\mathbf{CO}}(I_n, A, B, C) \leq \left( \|A_c\|_2 \sigma_{i+1}^{\mathbf{O}} + \min \left\{ \sigma_1^{\mathbf{CO}}, \sigma_1^{\mathbf{C}} \sigma_i^{\mathbf{O}} \right\} \right) \frac{1}{\sigma_i^{\mathbf{O}}} + \sigma_{i+1}^{\mathbf{O}}.$$

Let us consider the quadruple $(I_n, A' + \delta A', B' + \delta B', C' + \delta C')$ with

$$\delta A' = \begin{pmatrix} 0 & -A_2 \\ 0 & 0 \end{pmatrix}, \quad \delta B' = \begin{pmatrix} -B_1 \\ 0 \end{pmatrix}, \quad \delta C' = \begin{pmatrix} 0 & -C_2 \end{pmatrix}$$

with $A_2 \in M_{i \times (n-i)}(\mathbb{R}), B_1 \in M_{i \times m}(\mathbb{R}), C_2 \in M_{p \times (n-i)}(\mathbb{R})$. For all $i \in \{1, \ldots, n-1\}$, the quadruple $(I_n, A' + \delta A', B' + \delta B', C' + \delta C')$ is uncontrollable and unobservable and

$$\begin{aligned} \|(0, \delta A', \delta B', \delta C')\|_2 &\leq \|\delta A'\|_2 + \|\delta B'\|_2 + \|\delta C'\|_2 = \|A_2\|_2 + \|B_1\|_2 + \|C_2\|_2 \\ &\leq \left( \|A_c\|_2 \sigma_{i+1}^{\mathbf{O}} + \min \left\{ \sigma_1^{\mathbf{CO}}, \sigma_1^{\mathbf{C}} \sigma_i^{\mathbf{O}} \right\} \right) (\sigma_i^{\mathbf{O}})^{-1} + \sigma_{i+1}^{\mathbf{O}} \qquad \diamond \end{aligned}$$

**Remark.** If the quadruple $(I_n, A, B, C)$ is controllable and observable, so is the quadruple $(I_n, A^t, C^t, B^t)$. Applying Lemma 5 to this quadruple, we obtain that there exists, for all $i \in \{1, \ldots, n-1\}$, an orthogonal matrix $Q$ such that

$$A'' = Q A^t Q^t = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix}, \quad C'' = Q C^t = \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix}, \quad B'' = B^t Q^t = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{pmatrix}$$

where $\mathcal{A}_1 \in M_i(\mathbb{R})$, $\mathcal{B}_1 \in M_{i\times m}(\mathbb{R})$, $\mathcal{C}_1 \in M_{p\times i}(\mathbb{R})$ and

$$\|\mathcal{A}_2\|_2 \leq \|\mathcal{A}_c\|_2 \frac{\sigma_{i+1}^{\mathbf{c}}}{\sigma_i^{\mathbf{c}}}, \quad \|\mathcal{B}_2\|_2 \leq \sigma_{i+1}^{\mathbf{c}}, \quad \|\mathcal{C}_1\|_2 \leq \min\left\{\frac{\sigma_1^{\mathbf{co}}}{\sigma_i^{\mathbf{c}}}, \sigma_1^{\mathbf{o}}\right\}$$

Therefore, if $(I_n, A, B, C)$ is a controllable and observable quadruple, a tighter bound for the distance from this quadruple to the nearest uncontrollable and unobservable quadruple is given in the statement of next Theorem.

**Theorem 5.** *Let $(I_n, A, B, C)$ be a controllable and observable quadruple of matrices. A bound for the distance from this quadruple and the nearest uncontrollable and unobservable one is given by*

$$\min\left\{\mu_{2,\mathbb{R}}^{\mathbf{co}}(I_n, A, B, C), \tilde{\mu}_{2,\mathbb{R}}^{\mathbf{co}}(I_n, A, B, C)\right\}$$

*where*

$$\mu_{2,\mathbb{R}}^{\mathbf{co}}(I_n,A,B,C) = \min\left\{\left(\|A_c\|_2\sigma_2^{\mathbf{o}} + \min\left\{\sigma_1^{\mathbf{co}}, \sigma_1^{\mathbf{c}}\sigma_1^{\mathbf{o}}\right\}\right)\frac{1}{\sigma_1^{\mathbf{o}}} + \sigma_2^{\mathbf{o}}, \dots, \left(\|A_c\|_2\sigma_n^{\mathbf{o}} + \min\left\{\sigma_1^{\mathbf{co}}, \sigma_1^{\mathbf{c}}\sigma_{n-1}^{\mathbf{o}}\right\}\right)\frac{1}{\sigma_{n-1}^{\mathbf{o}}} + \sigma_n^{\mathbf{o}}\right\}$$

$$\tilde{\mu}_{2,\mathbb{R}}^{\mathbf{co}}(I_n,A,B,C) = \min\left\{\left(\|A_c\|_2\sigma_2^{\mathbf{c}} + \min\left\{\sigma_1^{\mathbf{co}}, \sigma_1^{\mathbf{c}}\sigma_1^{\mathbf{o}}\right\}\right)\frac{1}{\sigma_1^{\mathbf{c}}} + \sigma_2^{\mathbf{c}}, \dots, \left(\|A_c\|_2\sigma_n^{\mathbf{c}} + \min\left\{\sigma_1^{\mathbf{co}}, \sigma_1^{\mathbf{o}}\sigma_{n-1}^{\mathbf{c}}\right\}\right)\frac{1}{\sigma_{n-1}^{\mathbf{c}}} + \sigma_n^{\mathbf{c}}\right\}$$

*where $\sigma_i^{\mathbf{c}}$, where $\sigma_i^{\mathbf{o}}$, $\sigma_i^{\mathbf{co}}$ are the singular values of the controllability matrix of the pair $(A, B)$, of the observability matrix of the pair $(C, A)$ and of the controllability-observbility of the triple $(A, B, C)$, respectively, and $A_c$ is the companion matrix of $A$.*

## 9. Examples and concluding remarks

It was mentioned in Section §7 that when studying a qualitative property of a quadruple by means of an associated matrix it is not always true that a perturbation of this matrix corresponds to the associated matrix of a perturbed quadruple (see example in §7). We deal only with matrices corresponding to perturbed quadruples.

Besides, next examples show the improvements of the bounds obtained in the preceding Sections.

**Example 1.** Let us consider the quadruple of matrices $(I_n, A, B, 0)$ where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The smallest singular value of $T_2(I_n, A, B, 0)$ is

$$\sigma = 0.4450418679$$

Then

$$\frac{\sigma}{\sqrt{2n+m}} = 0.1682100151$$

The bound in [EEK97], using the matrix pencil $(A\ B) - \lambda(I_3\ 0)$, is 0.1075799057. That is to say, we have obtained a larger safety neighbourhood.

**Example 2.** Let us consider the quadruple of matrices $(I_n, A, B, 0)$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The smallest singular value of $T_2(I_n, A, B, 0)$ is

$$\sigma = 0.2175925528$$

Then

$$\frac{\sigma}{\sqrt{2n+m}} = 0.08224225457$$

The bound provided in [EEK97], using the matrix pencil $(A\ B) - \lambda(I_3\ 0)$, is 0.02976029930.

**Example 3.** Let us consider the quadruple of matrices $(I_n, A, B, 0)$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\mathbf{C}(A, B) = \begin{pmatrix} 0 & 3 & 15 \\ 0 & 0 & -3 \\ 1 & 4 & 16 \end{pmatrix},$$

the singular values of $\mathbf{C}(A, B)$ are:

$$\sigma_1^{\mathbf{c}} = 22.68945837, \quad \sigma_2^{\mathbf{c}} = 1.018190280, \quad \sigma_3^{\mathbf{c}} = 0.3895735536$$

and the companion matrix of $A$ is

$$A_c = \begin{pmatrix} 0 & 0 & 18 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}$$

Then

$$\|A_c\|_2 = 21.93916272,$$

$$\sigma_2^{\mathbf{c}} \left( \frac{\|A_c\|_2}{\sigma_1^{\mathbf{c}}} + 1 \right) = 2.0028711015, \quad \sigma_3^{\mathbf{c}} \left( \frac{\|A_c\|_2}{\sigma_2^{\mathbf{c}}} + 1 \right) = 8.783797845$$

Then

$$\mu_{2,\mathbb{R}}^{\mathbf{c}}(I_n, A, B, 0) = 2.002711015$$

Note that

$$\sigma_3^{\mathbf{C}}\left(\frac{\|A_c\|_2}{\sigma_2^{\mathbf{C}}}+1\right)=8.783797845>\mu_{2,\mathbb{R}}^{\mathbf{C}}(I_n,A,B,0)$$

**Example 4.** Let us consider the quadruple $(I_n, A, B, C) \in \mathcal{V}_1(\mathbb{R})$, where

$$A=\begin{pmatrix} -0.5 & -0.4 \\ 0 & -0.5 \end{pmatrix}, \quad B=\begin{pmatrix} 0 \\ 0.4 \end{pmatrix}$$

Then

$$\mathbf{C}(A,B)=\begin{pmatrix} 0 & -0.16 \\ 0.4 & -0.20 \end{pmatrix}$$

Its singular values are $\sigma_1^{\mathbf{C}} = 0.4535263685$ and $\sigma_2^{\mathbf{C}} = 0.1411163814$.

The matrix

$$X=\begin{pmatrix} 0 & 0 \\ 0.4 & -0.20 \end{pmatrix}$$

is a perturbed matrix of $\mathbf{C}(A, B)$ and its 2-distance to $\mathbf{C}(A, B)$ is: $d_2(\mathbf{C}(A, B), X) = 0.16$, which is approximately the value of the smallest singular value of $\mathbf{C}(A, B)$. The matrix $X$ is the controllability matrix of the quadruple $(I_n, A', B', 0)$, with

$$A'=\begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, \quad B'=\begin{pmatrix} 0 \\ 0.4 \end{pmatrix}$$

Then $d_2((I_n, A, B, 0), (I_n, A', B', 0)) = 0.4 > 0.16$, which is approximately the value of the largest singular value of $\mathbf{C}(A, B)$.

We conclude that the distance from the perturbation of the controllability matrix to the controllability matrix of another quadruple which is uncontrollable does not provide a good measurement for the distance from our controllable quadruple to another quadruple which is uncontrollable.

The computation of $\mu_{2,\mathbb{R}}^{\mathbf{C}}(I_n, A, B, 0)$ yields 0.58.

**Example 5.** Let us consider the quadruple $(I_n, A, B, C)$ where

$$A=\begin{pmatrix} 0.1 & 0.1 & 0 \\ 0 & 0.01 & 0.01 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad B=\begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix}, \quad C=\begin{pmatrix} 0.1 & 0 & 0 \end{pmatrix}$$

We are interested in obtaining bounds for $d_{2,\mathbb{R}}^{\mathbf{C}}(I_n, A, B, C)$, $d_{2,\mathbb{R}}^{\mathbf{O}}(I_n, A, B, C)$ and $d_{2,\mathbb{R}}^{\mathbf{CO}}(I_n, A, B, C)$. According to Theorems 2, 3 and 5 these bounds can be obtained after computing the singular values of the following matrices,

$$\mathbf{C}(A,B)=\begin{pmatrix} 0 & 0 & 0.0001 \\ 0 & 0.001 & 0.00002 \\ 0.1 & 0.001 & 0.00001 \end{pmatrix}$$

$$\mathbf{O}(C, A) = \begin{pmatrix} 0.1 & 0 & 0 \\ 0.01 & 0.01 & 0 \\ 0.001 & 0.0011 & 0.0001 \end{pmatrix}$$

$$\mathbf{CO}(A, B, C) = \begin{pmatrix} 0 & 0 & 0.00001 \\ 0 & 0.00001 & 0.0000012 \\ 0.00001 & 0.0000012 & 0.000000123 \end{pmatrix}$$

Straighforward computations yield the following bounds for the distance from this controllable and observable quadruple to the nearest one which is uncontrollable, unobservable, uncontrollable and unobservable, are, respectively:

$$\mu^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, C) = 0.01107294359$$

$$\mu^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C) = 0.01010136582$$

$$\min\{\mu^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C), \tilde{\mu}^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C)\} = 0.01118193072$$

**Example 6.** Let us consider the quadruple $(I_n, A, B, C)$ where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 10 & 0.1 & 0.01 \end{pmatrix}$$

As in Example 5, we compute the singular values of the controllability, observability and controllability-observability matrices of the triple $(A, B, C)$ which are, in this case,

$$\mathbf{C}(A, B) = I_3, \qquad \mathbf{O}(A, C) = \begin{pmatrix} 10 & 0.1 & 0.01 \\ 0.1 & 0.01 & 0 \\ 0.01 & 0 & 0 \end{pmatrix}, \qquad \mathbf{CO}(A, B, C) = \mathbf{O}(A, C)$$

The following bounds for the distance from this controllable and observable quadruple to the nearest one which is uncontrollable, unobservable, uncontrollable and unobservable, are obtained:

$$\begin{aligned} \mu^{\mathbf{c}}_{2,\mathbb{R}}(I_n, A, B, C) &= \\ \mu^{\mathbf{o}}_{2,\mathbb{R}}(I_n, A, B, C) &= \\ \min\{\mu^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C), \tilde{\mu}^{\mathbf{co}}_{2,\mathbb{R}}(I_n, A, B, C)\} &= 1.001245496 \end{aligned}$$

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